Magnetohydrodynamic flow of a viscous fluid past a sphere

By RICHARD VAN BLERKOM
International Business Machines Corporation

(Received 15 December 1959)

The flow of a viscous incompressible electrically conducting fluid past a sphere is studied; the uniform ambient flow field is colinear with the ambient uniform magnetic field. The force exerted on the sphere is computed for various conductivities and Reynolds numbers; of particular interest is the distinction in behaviour between the flow with ambient particle speed greater than ambient Alfvén speed and that with particle speed less than Alfvén speed.

1. Introduction

The flow of a viscous incompressible fluid past an obstacle at low Reynolds number has been the subject of many investigations. The rigorous analysis of this problem requires the solution of the non-linear Navier–Stokes differential equations. Stokes (1945), Oseen (1910), and Lewis & Carrier (1949) have formulated linear problems whose relevant solutions are good approximations to the observed physical facts. The linearized problems have been solved for flow past a sphere by Lamb (1945) and by Goldstein (1929) and the drag on the sphere has also been found. We shall use the same methods of linearization to study the flow of a viscous, incompressible, and electrically conducting fluid past an obstacle when the magnetic field and velocity are constant and parallel far from the obstacle. A detailed solution will be obtained for a sphere and a method of solution for other geometries will be indicated. The modification introduced in a uniform flow by an externally applied point force will be described.

2. The flow past a sphere

The magnetohydrodynamic flow of an incompressible viscous electrically conducting fluid of constant properties is governed by Maxwell's equations and the laws of conservation of mass and momentum. In m.k.s. units these equations take the form

$$\rho\left\{\frac{\partial \mathbf{V}'}{\partial t}+[\mathbf{V}\,.\,\nabla')\,\mathbf{V}'\right\} = -\nabla p' + \rho\nu\nabla^2\mathbf{V}' + \mu\mathbf{j}\times\mathbf{H}', \tag{2.1}$$

$$\operatorname{div}\mathbf{V}' = 0, \tag{2.2}$$

$$\operatorname{curl}\mathbf{H}' = \mathbf{j}, \quad \operatorname{div}\mathbf{H}' = 0, \tag{2.3}$$

$$\operatorname{curl}\mathbf{E} = 0. \tag{2.4}$$

These are supplemented by the constitutive relation

$$\mathbf{j} = \sigma[\mathbf{E} + \mu\mathbf{V}'\times\mathbf{H}']. \tag{2.5}$$

Here, $\mathbf{V}'$ is the fluid velocity, $\mathbf{H}'$ is the magnetic field, $\mathbf{E}$ is the electric field, σ is the electrical conductivity, ρ is the mass density, μ is the magnetic permeability, and ν is the kinematic viscosity. The differentiations are taken with respect to the physical variables x', y', z'.

The differential equations can be put in a dimensionless form by introducing the following substitutions:

$$\mathbf{V} = \frac{\mathbf{V}'}{U}, \quad p = \frac{a}{\rho\nu U}p', \quad x = \frac{x'}{a}, \quad y = \frac{y'}{a}, \quad z = \frac{z'}{a} \quad \text{and} \quad \mathbf{H} = \frac{\mathbf{H}'}{H_0}.$$

Here, U is a characteristic velocity of the problem, H_0 is a characteristic magnetic field, a is a characteristic length and the differentiations are with respect to x, y and z. In the problems we are considering U is the undisturbed velocity and H_0 is the magnetic field far from the obstacle. For steady-state problems, $\partial\mathbf{V}/\partial t = 0$ and (2.1) to (2.5) take the form

$$R(\mathbf{V}.\nabla)\mathbf{V} = -\nabla p + \nabla^2\mathbf{V} + \frac{\mu H_0 a^2}{\rho\nu U}\mathbf{j}\times\mathbf{H}, \tag{2.6}$$

$$\operatorname{div}\mathbf{V} = 0, \tag{2.7}$$

$$\operatorname{curl}\mathbf{H} = \frac{a}{H_0}\mathbf{j}, \quad \operatorname{div}\mathbf{H} = 0, \tag{2.8}$$

$$\operatorname{curl}\mathbf{E} = 0, \tag{2.9}$$

$$\mathbf{j} = \sigma\{\mathbf{E} + \mu U H_0 \mathbf{V}\times\mathbf{H}\}. \tag{2.10}$$

If we substitute (2.8) into (2.6) and use the vector identity

$$(\operatorname{curl}\mathbf{H})\times\mathbf{H} = (\mathbf{H}.\nabla)\mathbf{H} - \tfrac{1}{2}\nabla(\mathbf{H}.\mathbf{H}),$$

we obtain

$$R(\mathbf{V}.\nabla)\mathbf{V} = -\nabla p_0 + \nabla^2\mathbf{V} + \frac{M^2}{R_m}(\mathbf{H}.\nabla)\mathbf{H}, \tag{2.11}$$

where

$$p_0 = p + \frac{M^2}{2R_m}\mathbf{H}.\mathbf{H}. \tag{2.12}$$

Here, R_m is the magnetic Reynolds number given by $R_m = Ua\mu\sigma$ and M is the Hartmann number given by $M = \mu H_0 a(\sigma/\rho\nu)^{\frac{1}{2}}$. The ratio R_m/R is a measure of the relative importance of magnetic and viscous effects.

If we take the curl of (2.10) and make use of (2.7) and (2.9) and the vector identities

$$\operatorname{curl}(\operatorname{curl}\mathbf{H}) = \nabla(\operatorname{div}\mathbf{H}) - \nabla^2\mathbf{H} = -\nabla^2\mathbf{H} \tag{2.13a}$$

and

$$\operatorname{curl}(\mathbf{V}\times\mathbf{H}) = (\mathbf{H}.\nabla)\mathbf{V} - (\mathbf{V}.\nabla)\mathbf{H} + \mathbf{V}(\nabla.\mathbf{H}) - \mathbf{H}(\nabla.\mathbf{V}), \tag{2.13b}$$

we obtain

$$\nabla^2\mathbf{H} = -R_m\{(\mathbf{H}.\nabla)\mathbf{V} - (\mathbf{V}.\nabla)\mathbf{H}\}. \tag{2.14}$$

We shall confine our attention to problems in which the fluid flows past an axially symmetric obstacle in an otherwise unbounded domain with an ambient velocity $\mathbf{i}V_0$ and magnetic field $\mathbf{i}H_0$ which are uniform and directed along the symmetry axis of the object. Chester (1957) has treated this problem following a Stokes-like analysis; he computed the drag on a sphere to be

$$D = D_s\{1 + \tfrac{3}{8}M + \tfrac{7}{960}M^2 - \tfrac{43}{7680}M^3 + O(M^4)\}, \tag{2.15}$$

where

$$D_s = 6\pi\rho\nu a U. \tag{2.16}$$

We shall invoke an analysis similar to that of Oseen, although the motivation stems from the Carrier–Lewis approximation technique as used by Greenspan & Carrier (1959). In this technique one anticipates that the mathematical model retains significance when each of the undifferentiated quantities in the convective terms of equations (2.11) and (2.14) is replaced by an appropriate average which is taken to be some fraction of the free-stream values. In this paper, we shall take these functions to be unity, so that we shall treat a mathematical problem which is identical with that which would arise if the Oseen philosophy were followed. It should be noted, however, that the equations which would be obtained with a more appropriate choice of these averages would have solutions which are related to those obtained here by the formulas

$$\mathbf{V}_1(\mathbf{r}, C_1, C_2, C_3, R, R_m, \beta) = \mathbf{V}\left(\mathbf{r}, 1, 1, 1, \frac{R}{C_1}, \frac{R_m}{C_3}, \frac{C_1\beta}{C_2}\right), \tag{2.17a}$$

$$\mathbf{H}_1(\mathbf{r}, C_1, C_2, C_3, R, R_m, \beta) = \mathbf{H}\left(\mathbf{r}, 1, 1, 1, \frac{R}{C_1}, \frac{R_m}{C_3}, \frac{C_1\beta}{C_2}\right), \tag{2.17b}$$

where $C_1\mathbf{i}, C_2\mathbf{i}, C_3\mathbf{i}, C_4\mathbf{i}$ are the quantities which replace $\mathbf{V}, \mathbf{H}, \mathbf{H}, \mathbf{V}$, respectively, as they appear in (2.11) and (2.14).

With the foregoing, (2.11) and (2.14) become

$$R\frac{\partial \mathbf{V}}{\partial x} = -\nabla p_0 + \nabla^2\mathbf{V} + \frac{M^2}{R_m}\frac{\partial \mathbf{H}}{\partial x}, \tag{2.18}$$

$$\frac{1}{R_m}\nabla^2\mathbf{H} = \frac{\partial}{\partial x}(\mathbf{H} - \mathbf{V}). \tag{2.19}$$

Since the problem has been linearized superposition may be used; we write $\mathbf{V} = \mathbf{i} + \mathbf{v}$ and $\mathbf{H} = \mathbf{i} + \mathbf{h}$. The boundary conditions require that $\mathbf{v}$ and $\mathbf{h}$ tend to zero as the distance from the obstacle goes to infinity, that $\mathbf{v} = -\mathbf{i}$ at the obstacle surface, and that h_t and μh_n be continuous at the obstacle surface. Here, h_n and h_t are, respectively, the component of magnetic field that is normal to the obstacle and the component that is tangent to the obstacle.

Equations (2.18) and (2.19) can be simplified in the following way. We define

$$\mathbf{v}_1 = \mathbf{v} - \frac{\alpha_1}{R_m}\mathbf{h}, \tag{2.20}$$

$$\mathbf{v}_2 = \mathbf{v} - \frac{\alpha_2}{R_m}\mathbf{h}, \tag{2.21}$$

where the α_j are the roots of

$$\alpha^2 + (R - R_m)\alpha - M^2 = 0. \tag{2.22}$$

We then obtain

$$R_1\frac{\partial \mathbf{v}_1}{\partial x} = -\nabla p_0 + \nabla^2\mathbf{v}_1, \tag{2.23}$$

$$R_2\frac{\partial \mathbf{v}_2}{\partial x} = -\nabla p_0 + \nabla^2\mathbf{v}_2, \tag{2.24}$$

where $R_j = R + \alpha_j$. Each of these equations is identical with that encountered in the classical Oseen treatment of viscous fluid problems and our investigation will parallel closely that of Oseen.

It is easily seen that

$$\mathbf{v} = \frac{1}{R_2-R_1}[(R_2-R)\,\mathbf{v}_1-(R_1-R)\,\mathbf{v}_2], \tag{2.25}$$

$$\mathbf{h} = \frac{R_m}{R_2-R_1}[\mathbf{v}_1-\mathbf{v}_2], \tag{2.26}$$

$$\operatorname{div}\mathbf{v}_1 = \operatorname{div}\mathbf{v}_2 = 0, \tag{2.27}$$

and that the 'modified' Reynolds numbers R_i are the roots of

$$R_i^2-(R+R_m)\,R_i+(RR_m-M^2) = 0, \tag{2.28}$$

and are always real.

We define

$$\beta = \frac{\mu H_0^2}{\rho U^2} = \frac{M^2}{RR_m} \tag{2.29}$$

and note that, for $\beta < 1$, each of the 'modified' Reynolds numbers is positive, but for $\beta > 1$ there is a positive and a negative 'modified' Reynolds number. These two cases will be considered separately. When $\beta = 1$ the 'modified' Reynolds numbers are $R+R_m$ and zero.

It is convenient to follow Lamb (1945) and introduce the potential $\overline{\chi}_j$ such that

$$V_j = -\overline{\chi}_j\mathbf{i}+R_j^{-1}\operatorname{grad}\overline{\chi}_j+\operatorname{grad}\sum_{n=0}^{\infty}A_{n_j}\frac{\partial^n(r^{-1})}{\partial x^n}. \tag{2.30}$$

In order that (2.23) and (2.24) be satisfied it is necessary that

$$\left[\nabla^2-R_j\frac{\partial}{\partial x}\right]\overline{\chi}_j = 0. \tag{2.31}$$

Furthermore,

$$p_0 = -\sum_{n=0}^{\infty}A_{n_j}R_j^{-1}\frac{\partial^{n+1}(r^{-1})}{\partial x^{n+1}}. \tag{2.32}$$

The axially symmetric solution to (2.31) that vanishes at infinity is

$$\overline{\chi}_j = e^{\frac{1}{2}(R_j)x}\sum_{n=0}^{\infty}B_{jn}\,\chi_n(|R_1|\,r)\,P_n(\cos\theta), \tag{2.33}$$

where θ is the polar angle measured in spherical co-ordinates (r,θ,ω) and

$$\chi_n(x) = (2n+1)\left(\frac{\pi}{x}\right)^{\frac{1}{2}}K_{n+\frac{1}{2}}\left(\frac{x}{2}\right). \tag{2.34}$$

Here, K_n is the modified Bessel function of the second kind.

It follows from (2.8), (2.26), (2.30) and (2.31) that the current paths are circles which have their centres on the x-axis. From the axial symmetry of the problem, equation (2.10), and Kirchhoff's law, we deduce that $\mathbf{E} = 0$ everywhere in the fluid. Inside the spherical obstacle the electric and magnetic fields are governed by Maxwell's equations and the charge conservation equations which, in m.k.s. units, are given by

$$\left.\begin{aligned}\operatorname{curl}\mathbf{h}_i = \frac{a}{H_0}\mathbf{j}_i,\quad \operatorname{div}\mathbf{h}_i = 0,\quad \mathbf{j}_i = \sigma_i\mathbf{E}_i,\\ \operatorname{curl}\mathbf{E}_i = 0,\quad \operatorname{div}\sigma_i\mathbf{E}_i = 0.\end{aligned}\right\} \tag{2.35}$$

Since the electric field vanishes in the fluid and on the boundary of the sphere, it follows that $\mathbf{E}_i = 0$ and that

$$\mathbf{h}_i = -\operatorname{grad} \sum_{n=0}^{\infty} D_n r^n P_n(\cos\theta). \tag{2.36}$$

We now follow the method of Goldstein (1929) and write the boundary conditions at the surface of the sphere ($r = 1$) in the form

$$\frac{(n+1)(-1)^n}{n!}\frac{R_m}{R}A_n - \frac{1}{2(R_2-R_1)}\sum_{n=0}^{\infty}\sum_{j=1}^{2}(-1)^j(R_m-R_j)B_{jm}\sigma_{n,m}(R_j) = \begin{cases}0 & (n=0,2,3,\ldots),\\ 1 & (n=1),\end{cases} \tag{2.37}$$

$$\frac{(-1)^n}{n!}\frac{R_m}{R}A_n + \frac{1}{2(R_2-R_1)}\sum_{m=0}^{\infty}\sum_{j=1}^{2}(-1)^j(R_m-R_j)B_{jm}\tau_{n,m}(R_j) = \begin{cases}0 & (n=2,3,\ldots),\\ 1 & (n=1),\end{cases} \tag{2.38}$$

$$\frac{(-1)^n(n+1)}{n!}\frac{R_m}{R}A_n - \frac{R_m}{2(R_2-R_1)}\sum_{m=0}^{\infty}\sum_{j=1}^{2}(-1)^j B_{jm}\sigma_{n,m}(R_j) = -\frac{\mu_s}{\mu}nD_n - \delta_{1n} \quad (n=0,1,2,\ldots), \tag{2.39}$$

$$\frac{(-1)^n}{n!}\frac{R_m}{R}A_n + \frac{R_m}{2(R_2-R_1)}\sum_{m=0}^{\infty}(-1)^j B_{jn}\tau_{n,m}(R_j) = D_n + \delta_{1n} \quad (n=1,2,\ldots), \tag{2.40}$$

where

$$\sigma_{n,m}(R_1 r) = 2\{\chi_n'(|R_1|r)\psi_{n,m}(|R_1|r) - \chi_m(|R_1|r)\psi_{n,m}'(|R_1|r)\}\left(\frac{|R_1|}{R_1}\right)^n \tag{2.41}$$

and

$$\tau_{n,m}(R_1 r) = \frac{2\chi_m(|R_1|r)}{R_1}\{2X_{n,m}(|R_1|r) - \Psi_{n,m}(|R_1|r)\}\left(\frac{|R_1|}{R_1}\right)^n. \tag{2.42}$$

The functions $X_{n,m}$ and $\Psi_{n,m}$ are given by

$$\begin{aligned} X_{n,m} = {} & \frac{m(2n+1)}{n+1}\frac{(2m)!\,(2n)!}{(2n+2m)!}\left(\frac{(n+m)!}{(n)!\,(m)!}\right)^2 \\ & \times\left\{\frac{\psi_{n+m-1}}{\xi} + \frac{3(n-1)}{(2n-1)}\frac{m-1}{2m-1}\frac{2n+2m-1}{n+m}\frac{\psi_{n+m-3}}{\xi} + \frac{1.3.5}{2!}\frac{(n-1)(n-2)}{(2n-1)(2n-3)}\right. \\ & \times\frac{(m-1)(m-2)}{(2m-1)(2m-3)}\frac{(2n+2m-1)(2n+2m-3)}{(n+m)(n+m-1)}\frac{\psi_{n+m-5}}{\xi} \\ & +\ldots+\frac{1.3\ldots(2r+1)}{r!}\frac{(n-1)\ldots(n-r)}{(2n-1)\ldots(2n-2r+1)}\frac{(m-1)\ldots(m-r)}{(2m-1)\ldots(2m-2r+1)} \\ & \times\frac{(2n+2m-1)\ldots(2n+2m+2n+1)}{(n+m)\ldots(n+m-r+1)}\frac{\psi_{n+m-2r-1}}{\xi} \\ & \left.+\ldots+\frac{(n-1)\ldots(n-m+1)}{(2n-1)\ldots(2n-2m+3)}\frac{(2n+2m-1)\ldots(2n+3)}{(n+m)\ldots(n+2)}\frac{\psi_{n-m+1}}{\xi}\right\} \end{aligned} \tag{2.43}$$

and

$$\Psi_{n,m} = \frac{(2m)!\,(2n+1)!}{(2n+2m+1)!}\left(\frac{(n+m)!}{(n)!\,(m)!}\right)^2 \left\{\psi_{n+m} + \frac{n}{2n-1}\frac{m}{2m-1}\frac{2n+2m+1}{n+m}\psi_{n+m-2}\right.$$

$$+\frac{1.3}{2!}\frac{n(n-1)}{(2n-1)(2n-3)}\frac{m(m-1)}{(2m-1)(2m-3)}\frac{(2n+2m+1)(2n+2m-1)}{(n+m)(n+m-1)}\psi_{n+m-4}$$

$$+\ldots+\frac{1.3.5\ldots(2r-1)}{r!}\frac{n(n-1)\ldots(n-r+1)}{(2n-1)(2n-3)\ldots(2n-2r+1)}$$

$$\times\frac{m(m-1)\ldots(m-r+1)}{(2m-1)(2m-3)\ldots(2m-2r+1)}$$

$$\times\frac{(2n+2m+1)(2n+2m-1)\ldots(2n+2m-2r+3)}{(n+m)(n+m-1)\ldots(n+m-r+1)}\psi_{n+m-2r}$$

$$+\ldots+\frac{n(n-1)\ldots(n-m+1)}{(2n-1)(2n-3)\ldots(2n-2m+1)}$$

$$\left.\times\frac{(2n+2m+1)(2n+2m-1)\ldots(2n+3)}{(n+m)(n+m-1)\ldots n+1}\psi_{n-m}\right\}, \tag{2.44}$$

where

$$\psi_n(x) = (2n+1)\left(\frac{\pi}{x}\right)^{\frac{1}{2}} I_{n+\frac{1}{2}}\left(\frac{x}{2}\right). \tag{2.45}$$

Equation (2.37) is the condition $v_r = -\cos\theta$, equation (2.38) is the condition $v_\theta = \sin\theta$, equation (2.39) expresses the continuity of μh_r and equation (2.40) represents the continuity of h_θ. The infinite system of equations is approximated to order q by considering (2.37) for $n = 0, 1, \ldots, q$; (2.38) for $n = 1, \ldots, q$; (2.39) for $n = 0, 1, \ldots, q-1$; and (2.40) for $n = 1, 2, \ldots, q$. This yields $4q+1$ equations for the $4q+1$ unknowns $A_0, A_1, \ldots, A_q$; $B_{10}, B_{11}, \ldots, B_{1q-1}$; $B_{20}, B_{21}, \ldots, B_{2q-1}$; $D_1, D_2, \ldots, D_q$ with the rest of the A_n, B_{jn-1}, D_n taken as zero. Numerical results for the constants were obtained using a second-order approximation and several values of the 'modified' Reynolds number. For small 'modified' Reynolds numbers it was found that A_0, A_1, B_{10}, B_{20} and D_1 are all much greater than A_2, B_{11}, B_{21} and D_2.

The drag D on a solid obstacle is obtained by applying Newton's second law of motion to the fluid surrounding the obstacle and bounded by a surface S which is everywhere far from the obstacle. The total force exerted on the fluid within S is the sum of the viscous force exerted by the obstacle $(-D)$, the force exerted by the fluid outside S, plus the ponderomotive force.

We are able to show by direct calculation from the results of § 2 that for small 'modified' Reynolds numbers the ponderomotive force $\{\beta R(\operatorname{curl}\mathbf{H})\times\mathbf{H}\}$ is small compared with the viscous forces $\nu\nabla^2\mathbf{V}$ in all parts of the fluid provided that $\mu = \mu_s$. This restriction on the permeabilities is necessary so that near the sphere $\mathbf{H} \approx \mathbf{i}$. Hence in computing the drag we neglect the body force. This calculation can be carried out by an extension of the process used by Lamb in showing for small Reynolds number that $(\operatorname{curl}\mathbf{V})\times\mathbf{V}$ may be neglected compared with viscous forces in a non-conducting fluid.

We continue in a manner similar to that used by Goldstein (1929) and obtain

$$C_D = \frac{D}{\rho U^2 a^2} = -\frac{R_m}{R}(1-\beta)\int \frac{\partial \phi}{\partial n} dS. \tag{2.46}$$

For flow past a spherical object we obtain

$$C_D = \frac{4\pi R_m}{R}(1-\beta) A_0 \quad (\mu = \mu_s). \tag{2.47}$$

The first-order solution for small 'modified' Reynolds numbers yields

$$C_D \approx \frac{6\pi}{R}(1+\tfrac{3}{8}R+O(R_i^2)) \quad (\beta \leqslant 1), \tag{2.48a}$$

$$C_D \approx \frac{6\pi}{R}\left[1-\frac{3R}{8}\left(\frac{R_m-2R_m\beta-R}{|R_2-R_1|}\right)\right] \quad (\beta \geqslant 1), \tag{2.48b}$$

when $\mu = \mu_s$.

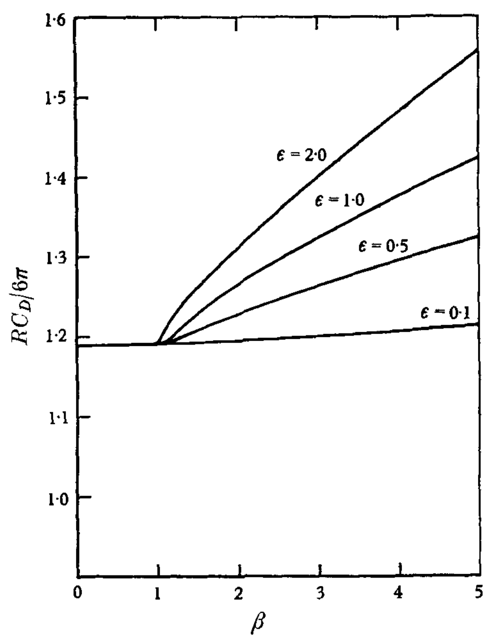

FIGURE 1. Drag coefficient *versus* β. $\epsilon = R_m/R$, $R = 0{\cdot}5$.

Numerical values were obtained for C_D at $R = \frac{1}{2}$ for several values of R_m and β, where all terms up to those of order R_i^2 were retained and it was found that (2.48*a*) and (2.48*b*) were changed by about 1 %. These results are plotted in figure 1. This does not mean that the error in these formulas for C_D is as small as 1 %, since this error is dependent on the accuracy of the linearization.

In order to obtain additional insight, the results of § 2 were used to obtain numerical values for the x-components of particle velocity and magnetic field along the lines $(0,0,z)$, $(5,0,z)$ and, for the subsonic case, $(-5,0,z)$, with $R = R_m = 0{\cdot}25$ and $\mu = \mu_s$. The results are shown in figure 2. It was found that both the particle velocity and magnetic field are independent of β for $0 \leqslant \beta \leqslant 1$.

In fact, in both the supersonic and the subsonic case ($\beta = 2$), the magnetic field was found to be small (less than 0·1) and hence was not plotted. If the sphere has a permeability different from that of the fluid, the magnetic field is distorted near the sphere but the effects are unimportant elsewhere.

As was mentioned previously, Chester (1957) extended the Stokes-type approximation for the flow past a sphere. This is equivalent to the linearization we have used, if the special case R and $R_m \to 0$, β large and $\mu = \mu_s$ is considered. This yields 'modified' Reynolds numbers of $\pm M$ and a drag, particle velocity and magnetic field which are all in agreement with Chester's results.

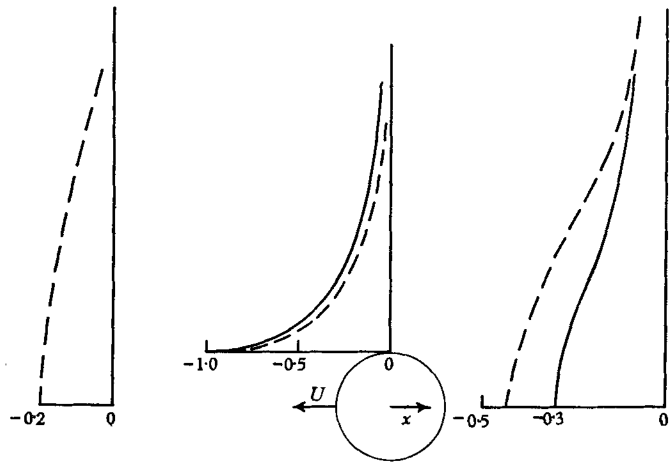

FIGURE 2. x-component of particle velocity *versus* position; $R = R_m = 0\cdot25$. The positions of the origins of the three graphs indicate the values of x at which $u(x, y)$ is plotted as a function of y. —, $\beta \leqslant 1$; ---, $\beta = 2$.

3. The point-force problem

We shall discuss here the flow field and magnetic field which arise when an externally applied point force acts on a fluid which otherwise would have uniform velocity and magnetic field. Such a description is of interest because it will be a major contribution to the description of the flow past objects such as that considered in § 2, and because it can be displayed in closed form with easy interpretation.

The governing equations are (2.14) and (2.11) modified only by the inclusion of the point-force contribution. That is,

$$R(\mathbf{V}.\nabla)\mathbf{V} + \nabla p_0 - \nabla^2\mathbf{V} + \frac{M^2}{R_m}(\mathbf{H}.\nabla)\mathbf{H} = -F\delta(r)\,\mathbf{i}, \tag{3.1}$$

with $$\delta(r) = \delta(x-0, y-0, z-0),$$

where $\delta(r)$ is the Dirac delta function and $-F\delta(r)\,\mathbf{i}$ is the force per unit volume applied at the origin. The equations of motion are put into dimensionless form and linearized, and linear combinations are taken in the same manner as in § 2. In this problem we define the characteristic length by

$$a = \sqrt{\frac{(6\pi\rho\nu U)}{F}}. \tag{3.2}$$

The exact solution to the linear problem is found to be

$$\mathbf{v}_j = -\frac{3}{2}\frac{\exp\left[-\frac{1}{2}|R_j|\,r+\frac{1}{2}(R_j)\,x\right]}{r}\mathbf{i}+\frac{3}{2R_j}\nabla\left(\frac{\exp\left[-\frac{1}{2}|R_j|\,r+\frac{1}{2}(R_j)\,x\right]}{r}\right)-\frac{3}{2R_j}\nabla\left(\frac{1}{r}\right), \tag{3.3}$$

and

$$p_0 = +\frac{3}{2}\frac{\partial}{\partial x}\left(\frac{1}{r}\right). \tag{3.4}$$

Expressions for $\mathbf{v}$ and $\mathbf{h}$ are obtained from (2.25) and (2.26). We note that at the origin the velocity becomes infinite as r^{-1} and is directed along the positive x-axis, while $\mathbf{h} = 0$.

At this point it is appropriate to compare our results for $\beta < 1$ with those for $\beta > 1$. The wake for the magnetohydrodynamic flow is the region in which $\exp\left[-\frac{1}{2}|R_j|\,r+\frac{1}{2}(R_j)\,x\right]$ is not small. In the wake the particle velocity and magnetic field vary as r^{-1} far from the origin, while in the region outside the wake they vary as r^{-2}. For $0 \leqslant \beta \leqslant 1$, the wake appears near the positive x-axis. However, for $\beta > 1$ the wake appears not only near the positive x-axis, but also near the negative x-axis.

We now find it convenient to turn our attention to the co-ordinate system in which the fluid is at rest at infinity. From (2.25), (2.26) and (3.3), it follows that far from the origin at points outside the wake for $\beta < 1$ the velocity and magnetic field are as if due to a source located at the origin and of strength $6\pi R_m/R_1R_2$. There is an equal inward flux along the wake near the x-axis. If $\beta > 1$, the magnetic field outside the wake is the same as that due to a sink at the origin, which is offset by concentrated outward flows near both the positive and the negative x-axes; the velocity is that which would be due to a sink equivalent to the magnetic field sink and, in addition, there is an inward flux near the positive x-axis, both of which are balanced by an outward flux along the negative x-axis. The results for $\beta = 1$ are obtained by considering what happens in the limit as $\beta \rightarrow 1-$ or $1+$. These limits yield the same conclusions. There is a net influx of particles near the positive x-axis which is balanced by a source at the origin. The essentially different behaviour for $\beta < 1$ and $\beta > 1$ may be explained by noting that, if $\beta = 1$,

$$U = \sqrt{\left(\frac{\mu}{\rho}\right)}H_0, \tag{3.5}$$

which is equal to the Alfvén wave speed. Hence, for $\beta < 1$ the free-stream velocity is greater than the Alfvén velocity and for $\beta > 1$ the free-stream velocity is less than the Alfvén speed.

4. The infinite conductivity case

We shall now discuss the flow past an obstacle of a fluid which has infinite conductivity. For $\sigma \rightarrow \infty$ ($R_m \rightarrow \infty$) a possible solution to (2.7), (2.11) and (2.14) is $\mathbf{h} = \mathbf{v}$, which reduces the problem to the form

$$\left[\nabla^2 - R(1-\beta)\frac{\partial}{\partial x}\right]\mathbf{v} = \operatorname{grad} p_0, \tag{4.1}$$

$$\nabla . \mathbf{v} = 0. \tag{4.2}$$

These equations are identical with the equations of motion for a non-conducting fluid flow with Reynolds number $R(1-\beta)$ and pressure p_0. For $\beta > 1$, we deduce that the disturbance velocity is equivalent to the disturbance velocity due to the flow of a non-conducting fluid which, far from the obstacle, is in the direction of the negative x-axis and which has a Reynolds number $R(\beta-1)$. Hence, if we can satisfy the boundary conditions on the velocity in the problem in which there is no magnetic field, then we can satisfy the boundary conditions on velocity in the magnetohydrodynamical problem. The magnetic field satisfies the same boundary conditions at the obstacle ($\mathbf{H} = 0$) as the velocity as long as the obstacle has the same permeability as the fluid. The magnetic field inside the sphere is zero and the electric field is everywhere equal to zero by the same argument as in the previous section. For $\beta = 1$, (4.9) reduces to the equation obtained by Stokes for a non-conducting fluid. Naturally the suitability of this model can only be determined by the degree of agreement with experimental results.

The work reported in this paper is an extract from a thesis submitted in April 1959 in partial fulfilment of the requirements for the degree of Ph.D. in applied mathematics at Harvard University.

REFERENCES

Chester, W. 1957 The effect of a magnetic field on Stokes flow in a conducting fluid. *J. Fluid Mech.* **3**, 304.

Goldstein, S. 1929 The forces on a solid body moving through viscous fluid and the steady flow of viscous fluid past a fixed spherical obstacle at small Reynolds numbers. *Proc. Roy. Soc.* A, **123**, 216.

Greenspan, H. P. & Carrier, G. F. 1959 The magnetohydrodynamic flow past a flat plate. *J. Fluid Mech.* **6**, 77.

Lamb, H. 1945 *Hydrodynamics*. New York: Dover.

Lewis, J. A. & Carrier, G. F. 1949 Some remarks on the flat plate boundary layer. *Quart. Appl. Math.* **7**, 228.

Oseen, C. W. 1910 Ueber die Stokes' sche formel, und uber eine verwandte aufgabe in der hydrodynamik. *Arkiv. f. math.* 29.

Stokes, G. G. 1945 On the theory of the internal friction in fluids in motion. *Trans. Camb. Phil. Soc.* **8**, 287.